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Generalization of matching extensions in graphs (III)

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ABSTRACT

Proposing them as a general framework, Liu and Yu (2001) [6] introduced (n, k, d)-graphs to unify the concepts of deficiency of matchings, n-factor-criticality and k-extendability. Let G be a graph and let n, k and d be non-negative integers such that $n + 2k + d + 2 \leqslant |V(G)|$ and $|V(G)| - n - d$ is even. If on deleting any n vertices from G the remaining subgraph H of G contains a k-matching and each k-matching can be extended to a defect-d matching in H, then G is called an *(n, k, d)-graph*. In this paper, we obtain more properties of (n, k, d)-graphs, in particular the recursive relations of (n, k, d)-graphs for distinct parameters n, k and d. Moreover, we provide a characterization for maximal non-(n, k, d)-graphs.

1. Introduction

All graphs considered in this paper are finite, connected, loopless and have no multiple edges. For the most part our notation and terminology follow those of Bondy and Murty [3].

Let G be a graph with vertex set $V(G)$, edge set $E(G)$ and minimum degree $\delta(G)$. A *matching* M of G is a subset of $E(G)$ such that any two edges of M have no vertices in common. A matching of k edges is called a *k-matching*. For a matching M, we use $V(M)$ to denote the vertices incident to the edges of M. Let d be a non-negative integer. A matching is called a *defect-d matching* if it covers exactly $|V(G)| - d$ vertices of G. Clearly, a defect-0 matching is a perfect matching. For a subset S of $V(G)$, we denote by $G[S]$ the subgraph of G induced by S and we write $G - S$ for $G[V(G) \setminus S]$. The number of odd components of G is denoted by $c_0(G)$. The *join* $G \vee H$ of two graphs G and H is a graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy \mid x \in V(G),\ y \in V(H)\}$. We denote the complement of G by $\overline{G}$. A set T is called an *n-set* if $|T| = n$. For two disjoint sets A and B of $V(G)$, we define $E(A, B) = \{xy\ :\ x \in A \text{ and } y \in B\} \cap E(G)$.

Let M be a matching of G. If there is a matching M' of G such that $M \subseteq M'$, we say that M can be extended to M' or M' is an *extension* of M. Suppose that G is a connected graph with perfect matchings. If each k-matching can be extended to a perfect matching in G, then G is called *k-extendable*. To avoid triviality, we require that $|V(G)| \geq 2k + 2$ for k-extendable graphs. This family of graphs was introduced by Plummer [9]. A graph G is called *n-factor-critical* if after deleting any n vertices the remaining subgraph of G has a perfect matching. This concept is introduced by Favaron [4] and Yu [10], independently, and is a generalization of the notions of the well-known factor-critical graphs and bicritical graphs, the cases of $n = 1$ and 2, respectively. In [8], Lou investigated the relationship between $2k$-factor-criticality and k-extendability.

Let G be a graph and let n, k and d be non-negative integers such that $|V(G)| \geq n + 2k + d + 2$ and $|V(G)| - n - d$ is even. If on deleting any n vertices from G the remaining subgraph of G contains a k-matching and each k-matching in the subgraph can be extended to a defect-d matching, then G is called an *(n, k, d)-graph*. This term was introduced by Liu and Yu [6] as

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a general framework for unifying the concepts of defect-d matchings, n-factor-criticality and k-extendability. In particular, $(n, 0, 0)$-graphs are exactly n-factor-critical graphs and $(0, k, 0)$-graphs are just the same as k-extendable graphs. In [5,6], the recursive relations were shown for distinct parameters n, k and d and the impact of adding or deleting an edge for $d \geq 0$ was discussed. In this paper, we continue the investigation of (n, k, d)-graphs and obtain more recursive relations.

A graph G is called a *maximal non-(n, k, d)-graph* if G is not an (n, k, d)-graph but $G \cup e$ is an (n, k, d)-graph for every edge $e \in E(\overline{G})$. In [1], Ananchuen et al. studied maximal non-k-factor-critical graphs and maximal non-k-extendable graphs; they also provided a characterization of these graphs. In the current paper, we generalize their criteria to obtain a characterization of maximal non-(n, k, d)-graphs.

2. Known results

A necessary and sufficient condition for a graph to have a defect-d matching was given by Berge [2].

Lemma 2.1 (*Berge [2]*). *Let G be a graph and d an integer such that $0 \leq d \leq |V(G)|$ and $|V(G)| \equiv d (\mathrm{mod}\ 2)$. Then G has a defect-d matching if and only if for any $S \subseteq V(G)$*

$$c_0(G - S) \leq |S| + d.$$

In [6], Liu and Yu showed the following sufficient and necessary conditions for (n, k, d)-graphs.

Lemma 2.2 (*Liu and Yu [6]*). *A graph G is an (n, k, d)-graph if and only if the following conditions hold:*

(a) *for any $S \subseteq V(G)$ such that $|S| \geq n$, then*

$$c_0(G - S) \leq |S| - n + d,$$

(b) *for any $S \subseteq V(G)$ such that $|S| \geq n + 2k$ and $G[S]$ contains a k-matching, then*

$$c_0(G - S) \leq |S| - n - 2k + d.$$

It is a natural to try to find recursive relations among the graphs with different parameters n, k and d. Below is one result of such an endeavor.

Lemma 2.3 (*Liu and Yu [6]*). *Every (n, k, d)-graph is also an (n', k', d)-graph, where $0 \leq n' \leq n, 0 \leq k' \leq k$ and $n' \equiv n (\mathrm{mod}\ 2)$.*

3. Main results

Following the study of recursive relations of the previous work, we continue to investigate the effect of various graphic operations on (n, k, d)-graphs and recursive relations. We start with the following lemma.

Lemma 3.1. *If G is an (n, k, d)-graph, then it is also an $(n - 2, k + 1, d)$-graph.*

Proof. First, note that G is an $(n-2, 0, d)$-graph by Lemma 2.3. Since $|V(G)| \geq n+2k+d+2$, for any $(n-2)$-set $S \subseteq V(G)$ there exist $(k + 1)$-matchings in subgraph $G - S$.

Suppose, to the contrary, that G is not an $(n - 2, k + 1, d)$-graph. Then, by the definition, there exist an $(n - 2)$-set $R \subseteq V(G)$ and a $(k + 1)$-matching M which cannot be extended to a defect-d matching of $G - R$. By Lemma 2.1 and parity, there exists a subset S_0 in $G - R - V(M)$ such that

$$c_0(G - R - V(M) - S_0) \geq |S_0| + d + 2.$$

Let $S = S_0 \cup R \cup V(M)$. Then $|S| = |S_0| + |R| + 2(k + 1) \geq n + 2k$ and $G[S]$ contains k-matchings, and

$$c_0(G - S) = c_0(G - S_0 - R - V(M)) \geq |S_0| + d + 2 = |S| - n - 2k + d + 2,$$

a contradiction to Lemma 2.2(b). □

Theorem 3.2. *A graph G is an $(n + 2, k - 1, d)$-graph if and only if G is an (n, k, d)-graph and $G \cup e$ is an (n, k, d)-graph, for any $e \in E(\overline{G})$.*

Proof. If G is an $(n + 2, k - 1, d)$-graph, by Lemma 3.1, then G is an (n, k, d)-graph.

We show that $G \cup e$ is an (n, k, d)-graph for any $e \in E(\overline{G})$. Otherwise, there exists an edge $e_1 \in E(\overline{G})$ such that $G' = G \cup \{e_1\}$ is not an (n, k, d)-graph. By Lemma 2.2, we consider two cases:

Case 1. There exists a subset $S_1 \subseteq V(G') = V(G)$ such that $|S_1| \geq n$ and $c_0(G' - S_1) \geq |S_1| - n + d + 2$. However,

$$c_0(G - S_1) \geq c_0(G' - S_1) \geq |S_1| - n + d + 2,$$

a contradiction to G being an (n, k, d)-graph and Lemma 2.2(a).

Case 2. There exists a subset $S_2 \subseteq V(G') = V(G)$, where $|S_2| \geq n + 2k$ and $G'[S_2]$ contains a k-matching M_2 such that

$$c_0(G' - S_2) \geq |S_2| - n - 2k + d + 2.$$

If $e_1 \notin M_2$, then $|S_2| \geq n + 2k$ and $G[S_2]$ contains the k-matching M_2, and $c_0(G - S_2) \geq c_0(G' - S_2) \geq |S_2| - n - 2k + d + 2$, a contradiction to G being an (n, k, d)-graph and Lemma 2.2(b). So $e_1 \in M_2$. Let $M_2' = M_2 - \{e_1\}$. Then $|S_2| \geq n + 2k = (n + 2) + 2(k - 1)$ and $G[S_2]$ contains the $(k - 1)$-matching M_2'. Moreover,

$$c_0(G - S_2) \geq c_0(G' - S_2) \geq |S_2| - n - 2k + d + 2 = |S_2| - (n + 2) - 2(k - 1) + d + 2,$$

a contradiction to G being an $(n + 2, k - 1, d)$-graph.

Next we prove the sufficiency. Suppose that G is not an $(n+2, k-1, d)$-graph. Then there exist an $(n+2)$-set $S_3 \subseteq V(G)$ and a $(k - 1)$-matching M_3 which cannot be extended to a defect-d matching of $G - S_3 - V(M_3)$. By Lemma 2.1, there exists a vertex set $R \subseteq V(G - S_3 - V(M_3))$ such that

$$c_0(G - S_3 - V(M_3) - R) \geq |R| + d + 2.$$

For any two vertices u, v of S_3, if $uv \in E(\overline{G})$, let $e_2 = uv$, $M_3' = M_3 \cup \{e_2\}$, and $S_3' = S_3 \setminus \{u, v\}$; then we have

$$c_0((G \cup e_2) - S_3' - V(M_3') - R) = c_0(G - S_3 - V(M_3) - R) \geq |R| + d + 2,$$

a contradiction to the fact that $G \cup e$ is an (n, k, d)-graph, for any $e \in E(\overline{G})$; if $uv \in E(G)$, then $|S_3'| = n$ and M_3' is a k-matching of G, and

$$c_0(G - S_3' - V(M_3') - R) = c_0(G - S_3 - V(M_3) - R) \geq |R| + d + 2,$$

a contradiction to G being an (n, k, d)-graph. $\square$

Applying Lemma 3.1, we have sufficient and necessary conditions for $(n + 2k, 0, d)$-graphs.

Theorem 3.3. *A graph G is an $(n + 2k, 0, d)$-graph if and only if G is an (n, k, d)-graph and for any edge set $D \subseteq E(\overline{G})$, $G \cup D$ is an (n, k, d)-graph.*

Proof. If G is an $(n + 2k, 0, d)$-graph, clearly $G \cup D$ is also an $(n + 2k, 0, d)$-graph. Applying Lemma 3.1 repeatedly, we see that $G \cup D$ is an (n, k, d)-graph.

On the other hand, suppose that G is not an $(n + 2k, 0, d)$-graph; by Lemma 2.2, there exists a subset S with $|S| \geq n + 2k$ such that

$$c_0(G - S) \geq |S| - (n + 2k) + d + 2.$$

Let $S = \{u_1, \ldots, u_h\}$, where $h \geq n + 2k$ and $G' = G \cup \{u_{2i-1}u_{2i} \mid i = 1, \ldots, k\}$. Then $G'[S]$ contains a k-matching and we have

$$c_0(G' - S) = c_0(G - S) \geq |S| - (n + 2k) + d + 2.$$

By Lemma 2.2(b), G' is not an (n, k, d)-graph, a contradiction. $\square$

Let $n = 0$ and $d = 0$; we have the next corollary.

Corollary 3.4 (*Lou [8]*). *A graph G of even order is $2k$-factor-critical if and only if:*

(a) *G is k-extendable; and*
(b) *for any edge set $D \subseteq E(\overline{G})$, $G \cup D$ is k-extendable.*

In [7], Liu and Yu present several results concerning $(n, k, 0)$-graphs and its subgraphs. In particular, they proved that if $G - V(e)$ is an $(n, k, 0)$-graph for each $e \in F$ (where F is a fixed 1-factor in G), then G is an $(n, k, 0)$-graph. We generalize this result for any $d \geq 0$ and $n \geq d + 2$.

Theorem 3.5. *Let F be a perfect matching of a connected graph G, where $|V(G)| \geq n + 2k + d + 4$ and $n \geq d + 2$. If subgraph $G - V(e)$ is an (n, k, d)-graph for each $e \in F$, then G is also an (n, k, d)-graph.*

Proof. Assume that F is a perfect matching of G such that $G - V(e)$ is an (n, k, d)-graph for each $e \in F$. To see the existence of k-matchings in the subgraphs, we show a claim.

Claim 1. *For any n-set $T \subseteq V(G)$, $G - T$ contains k-matchings.*

If $F \cap E(G - T) = \emptyset$, then there exists an edge $e = ab \in F$ such that $a \in T$ and $b \in V(G - T)$. Let $T' = T \setminus \{a\} \cup \{c\}$, where $c \in V(G) - T - \{b\}$. Then $|T'| = n$ and $F \cap E(G - T') = \{e\}$. By the assumption of the theorem, $G - V(e)$ is an (n, k, d)-graph. Hence, $G - V(e) - T'$ has a defect-d matching M_1. Since $|V(G)| \geq n + 2k + d + 4$, M_1 contains at least $k + 1$ edges. Therefore, $G - T$ contains k-matchings.

If $F \cap E(G - T) \neq \emptyset$, let $e = ab \in F \cap E(G - T)$; then $G - V(e)$ is an (n, k, d)-graph. So $G - V(e) - T$ contains k-matchings and thus $G - T$ contains k-matchings.

Suppose that G is not an (n, k, d)-graph; by the definition and Claim 1, there exists a vertex set R of order n in G and a k-matching M of $G - R$ such that $G - R - V(M)$ has no defect-d matchings. Let $G' = G - R - V(M)$; by Lemma 2.1 and parity, there exists a subset S in G' such that

$$c_0(G' - S) = c_0(G - R - V(M) - S) \geq |S| + d + 2. \tag{1}$$

Claim 2. *$F \cap E(G[R \cup S]) = F \cap M = F \cap E(V(M), R \cup S) = F \cap E(C_i) = F \cap E(S, V(C_i)) = \emptyset$ for all C_i, where C_i is an odd component of $G' - S$.*

If there exists an edge $e \in (F \cap E(R)) \cup (F \cap E(S))$, say $e \in F \cap E(R)$, then we have

$$c_0(G - V(e) - (R \setminus V(e)) - V(M) - S) = c_0(G' - S) \geq |S| + d + 2.$$

So $G - V(e)$ is not an $(n - 2, k, d)$-graph, a contradiction to $G - V(e)$ being an (n, k, d)-graph and Lemma 2.3.

If there exists an edge $e \in F \cap E(R, S)$, where $e = ab$, $a \in S$, $b \in R$, let $c \in C_i$, $R' = R \setminus \{b\} \cup \{c\}$, and $S' = S \setminus \{a\}$. Then we have

$$c_0(G - V(e) - R' - V(M) - S') \geq c_0(G' - S) - 1 \geq |S'| + d + 2.$$

Thus $G - V(e)$ is not an (n, k, d)-graph, a contradiction.

If there exists an edge $e \in F \cap M$, then we have

$$c_0(G - V(e) - R - V(M \setminus \{e\}) - S) = c_0(G' - S) \geq |S| + d + 2.$$

Thus $G - V(e)$ is not an $(n, k - 1, d)$-graph, a contradiction.

Suppose that $e \in F \cap E(V(M), R)$. Let $e = uv$ and $ua \in M$, where $u \in V(M)$ and $v \in R$. Let $R_1 = (R \setminus \{v\}) \cup \{a\}$ and $M'' = M \setminus \{ua\}$. Then

$$c_0(G - V(e) - R_1 - V(M'') - S) \geq |S| + d + 2.$$

Thus $G - V(e)$ is not an $(n, k - 1, d)$-graph, a contradiction.

Using the similar arguments, we may show that $e \notin E(S) \cup E(V(M), S) \cup (\cup_i E(C_i)) \cup E(S, V(C_i))$ for any $e \in F$.

Claim 3. *$G' - S$ has no even components.*

Otherwise, let D be an even component of $G' - S$ and $e = ab \in F$, $a \in V(D)$. If $b \in R$, choose a vertex $c \in V(D) \setminus \{a\}$, let $R_2 = R \setminus \{b\} \cup \{c\}$; then

$$c_0(G - V(e) - R_2 - V(M) - S) \geq c_0(G' - S) \geq |S| + d + 2.$$

Thus $G - V(e)$ is not an (n, k, d)-graph, a contradiction. For $b \in S$, we arrive at a contradiction with a similar argument. So we may assume that $b \in V(M)$. Let $bc \in M$. Set $S_1 = S \cup \{c\}$. Note that $G'[D \setminus \{a\}]$ contains at least one odd component. So we have

$$c_0(G - V(e) - R - V(M \setminus \{bc\}) - S_1) \geq |S_1| + d + 2.$$

Hence $G - V(e)$ is not an $(n, k - 1, d)$-graph, a contradiction.

Finally, if e is in the component D, then

$$c_0(G - V(e) - R - V(M) - S) \geq c_0(G' - S) \geq |S| + d + 2.$$

Thus $G - V(e)$ is not an (n, k, d)-graph, a contradiction again.

For any vertex $x \in S$, by Claim 2 x cannot be matched in perfect matching F to any other vertex in S or any vertex in $R \cup V(M)$ or any vertex in an odd component, so we conclude that $S = \emptyset$.

Claim 4. *$c_0(G' - S) = c_0(G') = d + 2$.*

By (1), we need only show that $c_0(G') \leq d + 2$. Otherwise, suppose $c_0(G') \geq d + 3$. If there exists an edge $e = ab \in F \cap E(R, C_i)$, where $a \in C_i$ and $b \in R$, we choose a vertex x from another odd component C_j and let $R_1 = R \setminus \{b\} \cup \{x\}$; then

$$c_0(G - V(e) - R_1 - V(M)) \geq c_0(G') - 2 \geq d + 1.$$

Thus $G - V(e)$ is not an (n, k, d)-graph, a contradiction. Next, we assume that all vertices in $\cup_i C_i$ are matched to $V(M)$. Consider the alternating path $P = c_i x_1 y_1 \ldots x_m y_m c_j$ of $F \cup M$ starting at C_i and ending at C_j. Let $e = c_i x_1 \in F$ and $M' = M \bigtriangleup (P \setminus \{e\})$. Then

$$c_0(G - V(e) - R - V(M')) \geq c_0(G') - 2 \geq d + 1,$$

a contradiction.

Now we proceed to the proof of the theorem.

Since $|V(G')| \geq d+4$ and $c_0(G') = d+2$, there exists one odd component of order at least 3. Moreover, as $n \geq d+2$, $c_0(G') = d+2$ and $F \cap (E(R, V(M)) \cup E(R)) = \emptyset$, there must exist an edge $e = ab \in F$ from R to an odd component C_i with $|C_i| \geq 3$, where $a \in C_i$ and $b \in R$. Since $|C_i| \geq 3$, choose a vertex $x \in C_i \setminus \{a\}$. Let $R_2 = R \setminus \{b\} \cup \{x\}$. Then

$$c_0(G - V(e) - R_2 - V(M)) \geq c_0(G') = d+2,$$

a contradiction.

We complete the proof. $\square$

In [5], Jin et al. proved the recursive relation for adding a vertex.

Theorem 3.6 (*Jin et al. [5]*). *Let G be an (n, k, d)-graph with $k > 0$ and $n > d$. Then $G \vee x$ is an $(n+1, k-1, d)$-graph for any vertex $x \notin V(G)$.*

Here we present an example to show that the condition $n > d$ is necessary.

For $k > 0$ and $n \leq d$, let $d = n + r$ for some $r \geq 0$. We consider a bipartite graph $H = K_{m,m+r}$, where $m \geq n+k$. Then H is an $(n, k, n+r)$-graph, but $H \vee x$ is not an $(n+1, k-1, n+r)$-graph.

4. Maximal non-(n, k, d)-graphs

In this section, we provide a characterization of maximal non-(n, k, d)-graphs, which is a generalization of the characterization of maximal non-k-factor-critical graphs in [1].

Theorem 4.1. *Let G be a connected graph of order p and n, k, d be positive integers with $p + n + d \equiv 0 \pmod 2$. Then G is a maximal non-(n, k, d)-graph if and only if*

$$G \cong K_{n+2k+s} \vee \left(\bigcup_{i=1}^{s+d+2} K_{2t_i+1} \right),$$

where s and t_i are non-negative integers with $\sum_{i=1}^{s+d+2} t_i = \frac{p-n-2k-d}{2} - s - 1$.

Proof. Let $H = K_{n+2k+s}$ and $G_i = K_{2t_i+1}$ for $1 \leq i \leq s+d+2$. Suppose that the theorem does not hold. That is, there exists an edge $e \in E(\overline{G})$ such that $G' = G \cup e$ is not an (n, k, d)-graph. Then e is an edge connecting G_i and G_j for some i and j.

By Lemma 2.2 and the parity argument, then either:

(a) there exists a subset S' in G' with $|S'| \geq n$ and $c_0(G' - S') \geq |S'| - n + d + 2$; or
(b) there exists a subset S' in G' such that $|S'| \geq n+2k$ and S' contains a k-matching satisfying $c_0(G'-S') \geq |S'|-n-2k+d+2$.

Clearly, $V(H) \subseteq S'$ and so S' contains a k-matching. Thus we need only consider (b). Hence we have $c_0(G' - S') \geq |S'| - n - 2k + d + 2 \geq |V(H)| - n - 2k + d + 2 \geq d + s + 2$. If $c_0(G' - S') = d + s + 2$, then $|S'| = n + 2k + s$ and so $S' = V(H)$. Therefore we have $c_0(G' - S') = d + s$, a contradiction. Hence we have $|S'| > n + 2k + s$ and then $c_0(G' - S') > d + s + 2$. But $G' - S'$ contains at most $s + d + 2$ odd components, a contradiction.

Now we prove the necessity. Since G is a maximal non-(n, k, d)-graph, for any n-subset R of $V(G)$ there exists a k-matching M in $G - R$. Let $G' = G - R - V(M)$. By Lemma 2.1 and parity, there exists a set S' in G' such that

$$c_0(G' - S') \geq |S'| + d + 2.$$

Let $C_1, C_2, \ldots, C_r$ be odd components in $G' - S'$ and $|S'| = s$. We show that $r = s + d + 2$. Otherwise, $r \geq s + d + 3$ and so $r \geq s + d + 4$ by parity. Let $e = c_1c_2$, where $c_1 \in V(C_1)$ and $c_2 \in V(C_2)$. Clearly, $(G \cup e) - (R \cup M \cup S')$ contains at least $s+d+2$ odd components, i.e., $G \cup e$ is not an (n, k, d)-graph, a contradiction to the fact that G is a maximal non-(n, k, d)-graph.

We next show that $G' - S'$ has no even components. Otherwise, assume that $G' - S'$ contains an even component D. Let $e = dc_1$, where $d \in D$ and $c_1 \in V(C_1)$, and consider $G \cup e$. Clearly, $(G \cup e) - (R \cup M \cup S')$ contains exactly $s + d + 2$ odd components since the components D and C_1 together with the edge e form an odd component of $G \cup e$. Thus $G \cup e$ is not an (n, k, d)-graph, a contradiction.

Finally we show that $G[R \cup M \cup S']$ is complete. Otherwise, there exist vertices x and y in $R \cup M \cup S'$ such that $e = xy \notin E(G)$. Consider $G \cup e$. Since $(G \cup e) - (R \cup M \cup S')$ contains exactly $s + 2 + d$ odd components, $G \cup e$ is not an (n, k, d)-graph, a contradiction. By a similar argument, it is easy to see that each C_i is complete for $1 \leq i \leq s + d + 2$. Furthermore, each vertex of C_i $(1 \leq i \leq s + d + 2)$ is adjacent to every vertex of $G[R \cup M \cup S']$.

Now, for $1 \leq i \leq s + d + 2$, let $|V(C_i)| = 2t_i + 1$ for some non-negative integer t_i. Then $p = |V(G)| = n + 2k + s + \sum_{i=1}^{s+d+2} |V(C_i)| = n + 2k + 2s + d + 2 + 2\sum_{i=1}^{s+d+2} t_i \geq n + 2k + 2s + d + 2$. Therefore, $0 \leq s \leq \frac{p-n-2k-d}{2} - 1$ and $\sum_{i=1}^{s+d+2} t_i = \frac{p-n-2k-d}{2} - s - 1$ are as required. This completes the proof of the theorem. $\square$

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